Graded rings and modules

Def: A graded ring is a ring R along w/ a direct sum decomposition

$$R = R_0 \oplus R_1 \oplus \cdots \quad \text{s.t. } R_i R_j \subseteq R_{i+j}.$$

The nonzero elts $f \in R_d$ are homogeneous of degree d. $I \subseteq R$ is a homogeneous ideal if it's generated by homogeneous elements.

Ex: $R = k[x_1, \dots, x_n] = S_0 \oplus S_1 \oplus \cdots$ where S_i is the k-vector space of homogeneous polynomials of deg d. i.e. $S_0 = k$, S_1 is gen. by $x_1, \dots, x_n$, S_2 gen by $x_1^2, x_1x_2, x_1x_3, \dots$ (as k-vector spaces).

Def: If $R = R_0 \oplus R_1 \oplus \cdots$ is a graded ring, then a graded R-module is a module

$$M = \bigoplus_{-\infty}^{\infty} M_i$$

s.t. $R_i M_j \subseteq M_{i+j}$.

A morphism of graded R-modules is an R-module homomorphism $\varphi : M \to N$ that preserves degree. i.e. $\varphi(M_d) \subseteq N_d$.

Ex: let $R = k[x_1,\dots,x_n]$ w/ standard grading.

1.) $I \subseteq R$ a homogeneous ideal, then R/I is a graded module w/ grading determined by $R \to R/I$.

2.) let $M = R$, with $M_{-1} = R_0$, $M_0 = R_1$, $M_i = R_{i+1}$.
We denote this $M = R(1)$, "R twisted by one".

More generally, $M = R(d)$ is defined as $M_i = R_{d+i}$.

Geom. context: A homogeneous ideal $I \subseteq k[x_0,\dots,x_n]$ defines a closed algebraic set X in projective space.
For $d >> 0$, $\dim_k(I_d)$ is the dimension of the space of homog. polynomials of deg d vanishing on X.

Def: let M be a f.g. graded module over $R = k[x_1,\dots,x_n]$ w/ standard grading. The <u>Hilbert function of M</u> is $H_M : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ and is defined

$$H_M(s) := \dim_k M_s$$

(the dims are finite since M is f.g. and thus Noetherian)
We can define this more generally for modules/k-algebras, but for now we focus on modules/ $k[x_1,\dots,x_n]$.

Ex: If $M = R = k[x_1, \ldots, x_n]$ w/ standard grading,

$$H_M(s) = \begin{cases} 0 & \text{for } s < 0 \\ \binom{s+n-1}{n-1}, & s \geq 0. \end{cases}$$

\# deg s monomials in n variables

Ex: $M = k[x,y]/(x^2, y^3)$

$$M = \underset{1}{M_0} \oplus \underset{x,\,y}{M_1} \oplus \underset{xy,\,y^2}{M_2} \oplus \underset{xy^2}{M_3}$$

So $H_M(s) = \begin{cases} 1 & \text{for } s = 0, 3 \\ 2 & \text{for } s = 1, 2 \\ 0 & \text{otherwise} \end{cases}$

We can also give non-standard grading to familiar rings / modules.

Ex: $M = \dfrac{k[x,y,z]}{(x^4, x^2y, y^2)}$, but $\deg x = 1$, $\deg y = 2$, $\deg z = 3$

$$M = M_0 \oplus M_1 \oplus M_2 \oplus M_3 \oplus M_4 \oplus M_5 \oplus \cdots$$

	M_0	M_1	M_2	M_3	M_4	M_5
k-gens:	k	x	x^2, y	x^3, xy, z	xz	x^2z, yz
dim:	1	1	2	3	1	2

Formula for $\dim_k M_i$?

$H_M(s)$ is <u>additive</u>. That is, if

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of graded modules, then

$$0 \to M'_d \to M_d \to M''_d \to 0$$

is an exact sequence of k-vector spaces, so

$$H_M(d) = H_{M'}(d) + H_{M''}(d)$$

For f.g. modules, as soon as s is sufficiently large, $H_M(s)$ behaves like a polynomial

Thm (Hilbert): If M is a finitely generated graded module over $k[x_0, \ldots, x_n]$ (standard grading), then $H_M(s)$ agrees w/ a polynomial $P_M(s)$ (of degree $\leq n$) for $s >> 0$.

$P_M(s)$ is called the Hilbert polynomial of M.

Pf Sketch: By induction on # of variables.

Base case: M is a finite dimensional k vector space, so $H_M(s) = 0$ for for $s >> 0$.

Now say $n \geq 0$. Consider the map $M \to M$ given by $m \mapsto x_n m$. This will increase the degree by one, so in order to make the map graded, we twist by -1, and get an exact sequence

$$0 \to N \to M(-1) \xrightarrow{x_n} M \longrightarrow M/x_nM \longrightarrow 0$$

$\uparrow$ kernel

Thus, for each s, we get

$$H_{M/x_nM}(s) - H_M(s) + \underbrace{H_{M(-1)}(s)}_{\substack{\shortparallel \\ H_M(s-1)}} - H_N(s) = 0$$

Notice that $x_n K = 0$ and $x_n(M/x_nM) = 0$. Thus, both are $k[x_0, \dots, x_n]/(x_n) \cong k[x_0, \dots, x_{n-1}]$ - modules, so by induction, they agree w/ a poly. for $s >> 0$.

Thus, so does $H_M(s) - H_M(s-1)$, and let $Q(s)$ be the corresponding polynomial. That is, $Q(s) = H(s) - H(s-1)$ for $s \geq s_0$. Thus (exer!), $H(s) = Q(s) + H(s-1)$ is also a polynomial (in fact, of degree one larger). $\square$

More geometric context:

A homogeneous ideal $I \subseteq k[x_0, \dots, x_n]$ defines a <u>projective algebraic set</u> $X \subseteq \mathbb{P}^n$.

Set $R = k[x_0, x_1, \dots, x_n]/I$, the homogeneous coordinate ring. Then

- $\deg P_R(s) = \dim X$.

- If $d = \deg P_R(s)$, then

$d!$ (initial coeff) = "degree" of X = # of pts intersecting a general plane of complementary dim

For certain modules M, the Riemann-Roch Theorem computes the Hilbert polynomial, and the "Chern classes" of the corresponding sheaf are encoded in the coeffs of the Hilb. poly.